Path Tracking Control for Dubin's Cars

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Abstract

The problem of driving a car along a given path is considered. In order to model a realistic road–following problem, the car is supposed to move forward only and to have bounds on the turning radius (Dubin's car). We propose a discontinuous control scheme on the angular velocity of the vehicle, based on the theory of sliding modes, that achieves the goal of tracking an unknown path relying on measurements of the current distance from the path and of the heading angle error. As usual with sliding–mode techniques, in the actual implementation discontinuous inputs can be replaced by smooth controls, with little performance degradation.

1 Introduction

The literature on planning and control techniques for nonholonomic vehicles has grown extensive in the recent few years.

The planning problem for nonholomic vehicles requires an approach based on a mix of techniques from conventional, holonomic planning and nonlinear systems theory. Besides the kinematic constraints imposed by nonholonomy, most often the additional constraint that the radius of curvature of the paths of the vehicle are lower bounded must be considered. This restriction makes the kinematic model more similar to real–world vehicles encountered in most applications. Some fundamental results in this area have a direct bearing to the work reported in this paper. In particular, it was shown that the kinematic model of a car that can drive both forwards and backwards with bounded curvature (but allowing cusps in the path), is locally controllable. A car that can only move forwards with curvature bounds is still controllable, although not locally. For this latter type of vehicle, Dubin [3] studied the shortest paths joining two arbitrary configurations (this particular vehicle model is often referred to as "Dubin's car"). Dubin's optimal paths are composed of line segments and arcs of circles of minimum radius. Reeds and Shepp [4] extended Dubin's results to a car that can reverse its motion.

The control problem is particularly challenging for nonholonomic systems, due to a theorem of Brockett [7] that bars the possibility of stabilizing a nonholonomic vehicle about a nonsingular configuration by any continuous time–invariant static feedback. Nonsmooth (see e.g. Sordalen and Egeland, [25], Aicardi et al., [32], Astolfi, [28], Guldner and Utkin, [22]), time–varying (viz. Samson, [26], McCloskey and Murray, [20], Sampei et al., [23]), and dynamic extension algorithms (cf. e.g. d'Andréa–Novel et al., [9], DeLuca and DiBenedetto, [10]), have been proposed to face the point–stabilization problem. An interesting comparative study of three of the above methods by simulation and experimental results has been presented recently by Andersen et al. [33]. For nonholonomic systems, the problem of tracking a trajectory or a path is simpler in principle than stabilizing to a point. Here, by "path" we refer to a curve (with some regularity requirements) in the plane were the robot moves; while a "trajectory" is a path with an associated time law (in other words, the robot is asked to be at a given point of the path at a given time). For an example of trajectory tracking controllers, see e.g. Walsh et al., [8]; and Sordalen and Canudas de Wit, [11], Sarkar et al., [16] for path tracking controllers.

In most part of the nonholonomic vehicle control literature, however, curvature bounds on the trajectories resulting from application of the control laws have not been considered. Although the work of Souères and Laumond [5], who mapped the whole configuration space of a Reeds and Shepp car in the optimal trajectories to a given goal configuration, could in principle be used to build a feedback law to stabilize the goal itself with bounded–curvature paths, a detailed

ana–lysis of such controller has not been presented so far. On the other hand, as already noted, most real–world vehicles have limited turning radii.

In this paper we consider the design of a control law for path tracking by a Dubin's car (we restrict to considering a kinematic unicycle in this paper). The further restriction that the vehicle only moves forward is motivated by the fact that, in practical road–following problems, vehicles maintain a positive, approximately constant, velocity (no backups on the highways!). We therefore assume that the forward velocity is given, and are only concerned with lateral stabilization to the path. The path shape is free (under some mild regularity restrictions), and it is not assumed that it is known a priori to the controller. We assume also that the only information available to the controller is the vehicle's lateral distance from the path, its heading angle error, and the sign of the curvature of the reference path (this seems to be consistent with state–of–the–art technology in sensors for automated vehicles).

As a result, we propose a variable–structure control law for vehicle orientation, that stabilizes the vehicle on the given path. The controller is designed according to sliding–mode techniques, and is therefore discontinuous in time. However, in practice it can be implemented in a smoothed version, that eliminates chattering and maintains good performance.

2 Problem formulation

Let the reference path $\gamma \subset \mathbb{R}^2$ be described as the trace of the parametrized curve

$$\hat{\mathbf{g}}(s) = \left(\hat{x}(s)\hat{y}(s)\right), \quad \text{with } s \in (0, 1),$$

with the natural orientation induced by increasing s. The following restrictions are imposed on the path:

A) $\hat{\mathbf{g}}(s)$ has continuous first derivative $\hat{\mathbf{g}}'(s)$ in $(0, 1)$. The second derivative $\hat{\mathbf{g}}''(s)$ has only a finite number of discontinuity points in $(0, 1)$, and changes its sign only a finite number of times in $(0, 1)$.

B) We assume that the curvature of the reference path is not smaller than that allowable for the vehicle. Let R denote the minimum turning radius of the vehicle, and $\hat{R}(s)$ the radius of curvature of $\hat{\mathbf{g}}(s)$. Set $\hat{R}(s)$ to $+\infty$ at the discontinuity points of $\hat{\mathbf{g}}''$. Otherwise, set

$$\frac{1}{\hat{R}(s)} = \frac{\hat{x}'\hat{y}'' - \hat{y}'\hat{x}''}{\left(\hat{x}'^2 + \hat{y}'^2\right)^{3/2}}.$$

and also set $\hat{R}(s) = +\infty$ at inflexional points where $\hat{x}'\hat{y}'' - \hat{y}'\hat{x}'' = 0$. The assumption is then
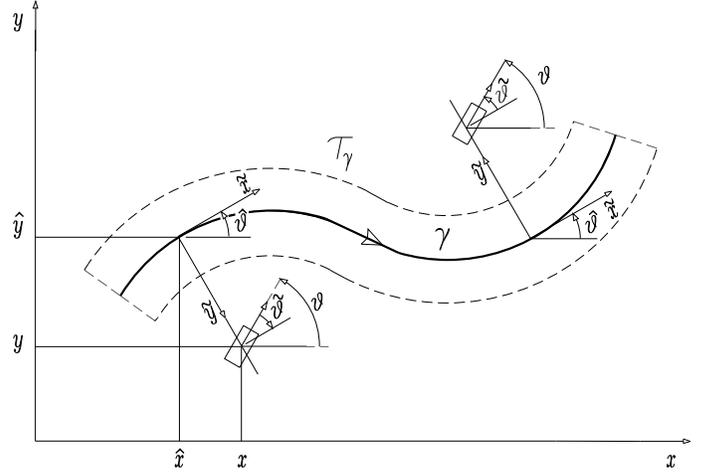

Figure 1: Reference path and coordinates associated with the configurations of the vehicle.

restated as

$$|\hat{R}(s)| \geq R, \quad s \in (0, 1). \tag{1}$$

C) We assume that the reference path does not come too close to itself. Consider an open neighbourhood of the path of radius R,

$$\mathcal{T}_\gamma = \left\{\mathbf{x} \in \mathbb{R}^2 : \exists s \in (0, 1), \|\mathbf{x} - \hat{\mathbf{g}}(s)\| < R\right\} \subset \mathbb{R}^2$$

where $\|\cdot\|$ denotes the euclidean norm. Then for all $\mathbf{x} \in \mathcal{T}_\gamma$ there exists a unique nearest point on $\hat{\mathbf{g}}$, say at $\bar{s} \in (0, 1)$, such that

$$\|\mathbf{x} - \hat{\mathbf{g}}(s)\| > \|\mathbf{x} - \hat{\mathbf{g}}(\bar{s})\|, \quad \forall s \neq \bar{s}.$$

The kinematic equations of the Dubin's car are written as

$$\begin{cases} \dot{x} &= \cos(\theta)\,u \\ \dot{y} &= \sin(\theta)\,u \\ \dot{\theta} &= \omega \end{cases} \tag{2}$$

with initial conditions (x_0, y_0, θ_0). Here, x, y, θ denote the posture of the unicycle with respect to the world frame and u, ω are the linear and angular velocity. In the classical Dubin's car, the forward velocity is fixed $u = \text{cost} > 0$. We adhere to this assumption for simplicity, although the generalization to the case $u(t) > 0$, $\forall t > 0$ is straightforward (see a related remark below). The only available control input is therefore the angular velocity ω.

In order to formalize and solve the control task of steering ω so as to converge to the desired path

and track it (with given velocity u), it is expedient to introduce a different set of coordinates for the state space. The path is embedded in a three-dimensional space, and consider the canonical frame $\mathcal{S}_T(s)$ associated with $\hat{\mathbf{g}}(s) = (\hat{x}(s), \hat{y}(s), 0)^T \in \mathbb{R}^3$. Recall that the canonical frame for a curve is defined by the tangent, the principal normal and the binormal of the curve at each point. In our case, the tangent and principal normal of $\mathcal{S}_T$ remain within to the plane where the car moves, while the binormal points upwards or downwards, depending on the local curvature, i.e., on $\text{sign}(\hat{R}(s))$.

Let $\hat{\theta}(s)$ denote the orientation of the tangent of the curve with respect to the x axis of a fixed frame

$$\hat{\theta}(s) = \text{atan2}\ (\hat{y}', \hat{y}').$$

Note that by assumption **A**) on $\hat{\mathbf{g}}(s)$, $\hat{\theta}$ is a continuous function in terms of s. Denote by $(\tilde{x}, \tilde{y}, \tilde{\theta})$ the configuration of the vehicle with respect to $\mathcal{S}_T$ (see fig.1). Notice that the positive sense of $\tilde{\theta}$ is taken according to the local orientation of the binormal axis. From elementary geometry we get

$$
\begin{aligned}
\tilde{x}(x, y, s) \ &= \ (x - \hat{x}(s))\cos(\hat{\theta}(s)) \\
&\quad + (y - \hat{y}(s))\sin(\hat{\theta}(s)),
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\tilde{y}(x, y, s) \ &= \ \text{sign}(\hat{R}(s))\left[(y - \hat{y}(s))\cos(\hat{\theta}(s)) \right. \\
&\quad \left. - (x - \hat{x}(s))\sin(\hat{\theta}(s))\right],
\end{aligned}
\tag{4}
$$

$$
\tilde{\theta}(\theta, s) \ = \ \text{sign}(\hat{R}(s))(\theta - \hat{\theta}(s)).
\tag{5}
$$

Based on assumption **C**), it is possible to associate to every point (x, y) of the neighborhood $\mathcal{T}_\gamma$ of the path, a unique frame $\mathcal{S}_T(\bar{s})$ with origin in the point of the path closest to (x, y). In fact, an application $\bar{s} : \mathcal{T}_\gamma \subset \mathbb{R}^2 \to (0, 1) \in \mathbb{R}$ is implicitly defined through (3) as

$$\tilde{x}(x, y, \bar{s}) = 0 \tag{6}$$

By our assumptions on the path, it also follows that $\bar{s}(x, y)$ is continuous everywhere, and differentiable almost everywhere, on $\mathcal{T}_\gamma$.

Consider the change of coordinates

$$
\mathcal{M} : \left(\begin{array}{c} x \\ y \\ \theta \end{array}\right) \to \left(\begin{array}{c} \bar{s} \\ \tilde{y}|_{\bar{s}} \\ \tilde{\theta}|_{\bar{s}} \end{array}\right),
\tag{7}
$$

on the domain $\mathcal{T}_\gamma \times \mathbb{R}$ by means of (6), (4), and (5). The new coordinates $\tilde{y}$ and $\tilde{\theta}$ are the lateral distance from the path and the heading angle error of the vehicle, i.e., they represent a natural choice for describing a road–following task. The change of coordinates
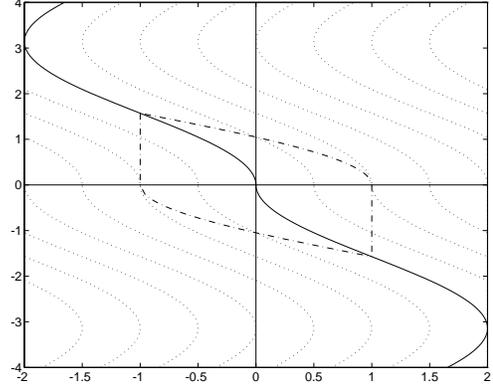

Figure 2: Sliding manifold (solid line), level surfaces of the σ function (dotted lines), and domain of attractiveness in the reduced state space.

is legitimate as it is injective on its domain. Notice however that this change of coordinates not only is not a diffeomorphism (as changes of coordinates are in nonlinear control theory usually), but it is not even continuous. In fact, due to the presence of the term $\text{sign}(\hat{R}(s))$ in (4), (5), a change of curvature along the path produces a jump of the variables $\tilde{y}, \tilde{\theta}$ to the symmetric point with respect to the origin in the $\tilde{y}, \tilde{\theta}$ plane (see fig.2). The introduction of discontinuous changes of coordinates have been used previously in the nonholonomic vehicle literature, e.g. by Aicardi *et al.* [32]. In that case, the new coordinate set (along with an input transformation) allowed authors to design smooth asymptotic point stabilizers without violating Brockett's theorem. Astolfi [28] independently developed a similar technique, and reported about the similarity of these coordinate changes with Arnold's "sigma" method for studying ODE's (Arnold [6]).

By differentiating (6), (4) and (5), applying the implicit function theorem, and using (2), the dynamic equations in the new state space (whose coordinates will henceforth be denoted by $(s, \tilde{y}, \tilde{\theta})^T$ for notational

convenience) are derived as

$$
\left\{
\begin{array}{rcl}
\dot{s} & = & \dfrac{\cos(\tilde{\theta})}{\cos(\tilde{\theta})\hat{x}' + \sin(\tilde{\theta})\hat{y}' - \tilde{y}\theta'}\, u \\[2ex]
\dot{\tilde{y}} & = & \sin(\tilde{\theta})\, u + 2\delta(\hat{R}(s))\, \mathrm{sign}(\hat{R}(s))\tilde{y} \\[2ex]
\dot{\tilde{\theta}} & = & -\dfrac{\cos(\tilde{\theta})}{|\hat{R}(s)| - \tilde{y}}\, u + \mathrm{sign}(\hat{R}(s))\big[\omega \\[2ex]
 & & \qquad + 2\delta(\hat{R}(s))\, \mathrm{sign}(\hat{R}(s))\tilde{\theta}\big]
\end{array}
\right.
\tag{8}
$$

For we assume that the path is not known a priori, the geometric parameters $\hat{x}'(s)$, $\hat{y}'(s)$, $\hat{\theta}'(s)$, appearing in the first state equation are not available to the controller. It should also be noted that, even if the path were known in advance, computations would be extremely awkward for all but the simplest path shapes. This is one reason why most path tracking controllers in the literature assume that reference paths are comprised of straight lines and circles only.

To overcome this difficulty, we consider a reduced state space $(\tilde{y}, \tilde{\theta})^T$, and regard the action of $\hat{R}(s)$ as a disturbance, about which the only available information is its sign and the lower bound given in assumption **B)**. The origin of the reduced state space corresponds to a motion of the vehicle along the desired path, with velocity u. Therefore, our control problem can be formulated as follows:

Problem 1.

Find a feedback control law $\omega(\tilde{y}, \tilde{\theta},\, \mathrm{sign}(\hat{R}), u)$ satisfying the curvature constraint

$$
\left| \frac{\omega}{u} \right| \leq \frac{1}{R},
\tag{9}
$$

such that, for any initial configuration $(x_0,\, y_0,\, \theta_0)$ of the vehicle in a suitable neighborhood of the path $\mathcal{D}_\gamma$, $(\tilde{y}, \tilde{\theta})$ converge to zero, irrespective of bounded disturbances of unknown amplitude.

3 Variable Structure Control

As mentioned in the problem formulation, we look for a controller that makes the path attractive for all states in a region near the path itself. Note that this is not a very restrictive condition in practice, as it is reasonable that, when the vehicle is very far from the desired path, a further instance of the planning algorithm is invoked for generating a new reference path.

Consider an open neighborhood $\mathcal{D}_\gamma$ of the path in the reduced state space as

$$
\mathcal{D}_\gamma = \Big\{ (\tilde{y}, \tilde{\theta}) : \tilde{y} < R,
$$

$$
-\arccos\left(\frac{1}{2} - \frac{\tilde{y}}{2R}\right) < \tilde{\theta} < \arccos\left(\frac{1}{2} + \frac{\tilde{y}}{2R}\right) \Big\},
\tag{10}
$$

Notice that $\mathcal{M}^{-1}(\mathcal{D}_\gamma \times (0,1)) \subset \mathcal{T}_\gamma \times (-\pi/2, \pi/2)$. Observe also that $\mathcal{D}_\gamma$ is symmetric with respect to the origin of the plane $(\tilde{y},\ \tilde{\theta})$ (see fig.2). Therefore, states within $\mathcal{D}_\gamma$ remain inside $\mathcal{D}_\gamma$ after any change of the sign of $\hat{R}(t)$. Since, according to assumption **A)** in section 2, there are only a finite number of such jumps, in our discussion we consider the evolution of states corresponding to the open intervals where $\hat{R}(t)$ does not change sign, and consider the effects of jumps separately.

Our proposed controller is based on the so-called sliding–mode design technique (see e.g. Utkin [1]). Let us introduce a sliding manifold in the reduced state space as

$$
\sigma(\tilde{y}, \tilde{\theta}) = 0,
\tag{11}
$$

where

$$
\sigma(\tilde{y}, \tilde{\theta}) = -\frac{\tilde{y}}{R} - \mathrm{sign}(\tilde{\theta})(1 - \cos(\tilde{\theta})).
\tag{12}
$$

Note that the function $\sigma(\tilde{y}, \tilde{\theta})$ is continuously differentiable once with respect to $\tilde{\theta}$, with $\frac{\partial \sigma}{\partial \tilde{\theta}}|_{\tilde{y},0} = 0$.

In the sliding mode control literature, the "equivalent control" is the input signal that causes a motion in the state space on the sliding surface $\sigma = 0$. The equivalent control is found by solving the equation $\dot{\sigma}(t) = 0$ in terms of the unknown control input. By differentiating (12)

$$
\dot{\sigma} = -\sin(\tilde{\theta})\left(\frac{u}{R} + \mathrm{sign}(\hat{R}(t))\, \mathrm{sign}(\tilde{\theta})\omega + \right.
$$
$$
\left. - \mathrm{sign}(\tilde{\theta})\frac{\cos(\tilde{\theta})}{|\hat{R}(t)| - \tilde{y}}\, u \right),
\tag{13}
$$

the equivalent control ω_{eq} is derived as

$$
\omega_{eq} = -\mathrm{sign}(\hat{R}(t))\left(\mathrm{sign}(\tilde{\theta}) - \cos(\tilde{\theta})\frac{R}{|\hat{R}| - \tilde{y}} \right)\frac{u}{R}.
\tag{14}
$$

It should be noted that ω_{eq} does not satisfy the constraint (9) on the minimum radius of curvature for negative values of $\tilde{\theta}$.

The dynamics of motion along the sliding manifold are studied by replacing (14) in (8), to get

$$
\left\{
\begin{array}{rcl}
\dot{\tilde{y}} & = & \sin(\tilde{\theta})\, u \\[1ex]
\dot{\tilde{\theta}} & = & -\mathrm{sign}(\tilde{\theta})\frac{u}{R}
\end{array}
\right.
\tag{15}
$$

Starting from any initial state $\tilde{y}_0 \neq 0$, $\tilde{\theta}_0 \neq 0$ on the sliding manifold, $|\tilde{\theta}(t)|$ monotonically decreases until

zero is reached in finite time $\tilde{\theta}_0 R/u$. From the definition of the sliding manifold (12), $\tilde{\theta} = 0 \Rightarrow \tilde{y} = 0$. A sliding regime on $\sigma = 0$ therefore implies convergence of the states to the origin of the reduced state space, hence perfect path tracking.

As already noticed, however, the equivalent control is not feasible by our Dubin's car in the region $\tilde{\theta} < 0$. Therefore, our next goal is to design a feasible control ω that guarantees attractivity of the feasible portion of the sliding surface $\sigma = 0$, $\tilde{\theta} \geq 0$. To this purpose, consider the following control law:

$$\omega = \text{sign}(\hat{R}(t)) \, \text{sign}(\sigma) \, \frac{u}{R}. \qquad (16)$$

The corresponding closed loop equations are written as

$$\begin{cases} \dot{\tilde{y}} &= \sin(\tilde{\theta}) \, u \\ \dot{\tilde{\theta}} &= \left(\text{sign}(\sigma) - \cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \right) \frac{u}{R} \end{cases} , \qquad (17)$$

and, plugging (16) into (13), we get

$$\dot{\sigma} = -\sin(\tilde{\theta}) \left(1 + \text{sign}(\tilde{\theta}) \, \text{sign}(\sigma) \right. \qquad (18)$$

$$\left. - \text{sign}(\tilde{\theta}) \cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \right) \frac{u}{R}.$$

The discussion is now split in four cases, corresponding to a four–fold partition of the neighbourhood $\mathcal{D}_\gamma$:

Region 1: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma > 0, \text{ and } \tilde{\theta} > 0 \right\}$.

From (12), we get $\tilde{y} < 0$ and hence

$$0 < \frac{R}{|\hat{R}(t)| - \tilde{y}} < 1.$$

From (17), $\dot{\tilde{\theta}} > 0$ and $\dot{\tilde{y}} > 0$. From (12), $\tilde{\theta} < \frac{\pi}{2}$ holds as long as $\sigma > 0$, hence in this sector it holds

$$\sigma \dot{\sigma} = -\sigma \sin(\tilde{\theta}) \left(2 - \cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \right) \frac{u}{R}$$

$$< -|\sigma| \, |\sin(\tilde{\theta})| \frac{u}{R} < 0.$$

If a change of the sign of the curvature occurs at any time at which the state is within region 1, the state jumps istantaneously to the symmetric point in region 2.

Region 2: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma < 0, \text{ and } \tilde{\theta} < 0 \right\}$.

From (17), $\dot{\tilde{\theta}} < 0$ and $\dot{\tilde{y}} < 0$. Hence $\tilde{y} < R$ and $\tilde{\theta} > -\frac{\pi}{2}$ holds as long as $\sigma < 0$. Then we have

$$\sigma \dot{\sigma} = -\sigma \sin(\tilde{\theta}) \left(2 + \cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \right) \frac{u}{R}$$

$$< -2 \, |\sigma| \, |\sin(\tilde{\theta})| \frac{u}{R} < 0.$$

If a change of the sign of the curvature occurs at any time at which the state is within region 2, the state jumps istantaneously to the symmetric point in region 1.

Region 3: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma < 0, \text{ and } \tilde{\theta} \geq 0 \right\}$.

Consider the function

$$\Gamma(\tilde{y}, \tilde{\theta}) = -1 - \frac{\tilde{y}}{R} + 2\cos\tilde{\theta}$$

and observe that, within region 3, $\Gamma(\tilde{y}, \tilde{\theta}) > 0 \Leftrightarrow (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma$. The derivative of Γ along the state trajectories is calculated as

$$\dot{\Gamma} = \sin(\tilde{\theta}) \frac{u}{R} + 2\sin(\tilde{\theta}) \cos(\tilde{\theta}) \frac{u}{|\hat{R}(t)| - \tilde{y}}.$$

It can be easily checked that $\dot{\Gamma}(\tilde{y}, \tilde{\theta}) > 0$ for all states within this region, except for $\tilde{\theta} = 0$. Any trajectory within region 3 cannot escape the region from the boundary $\Gamma = 0$, and will either reach the sliding surface $\sigma = 0$ (in the $\tilde{\theta} > 0$ branch), pass to region 2 through the boundary $\tilde{\theta} = 0$, or jump symmetrically to region 4 for a change of sign of $\hat{R}(t)$. Moreover, the sliding manifold is attractive for trajectories in this region:

$$\sigma \dot{\sigma} = \sigma \sin(\tilde{\theta}) \cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \frac{u}{R}$$

$$= -|\sigma| \, |\sin(\tilde{\theta})| \, |\cos(\tilde{\theta})| \frac{u}{|\hat{R}(t)| - \tilde{y}} < 0.$$

Region 4: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma > 0, \text{ and } \tilde{\theta} \leq 0 \right\}$.

Consider the function

$$\Gamma(\tilde{y}, \tilde{\theta}) = -1 + \frac{\tilde{y}}{R} + 2\cos\tilde{\theta}. \qquad (19)$$

Within region 4, we have that $\Gamma(\tilde{y}, \tilde{\theta}) > 0 \Leftrightarrow (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma$ and $\Gamma(\tilde{y}, \tilde{\theta}) < 1$. The derivative of Γ along the state trajectories is calculated as

$$\dot{\Gamma} = -\sin(\tilde{\theta}) \left(1 - 2\cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \right) \frac{u}{R}.$$

We have

$$\dot{\Gamma}(\tilde{y}, \tilde{\theta}) > 0 \Leftrightarrow 0 < \Gamma(\tilde{y}, \tilde{\theta}) < \frac{|\hat{R}(t)|}{R} - 1 \qquad (20)$$

except for $\tilde{\theta} = 0$.

It can be easily proved that starting within region 4 the trajectory cannot escape from the

boundary $\Gamma = 0$. In fact $\Gamma(\tilde{y}, \tilde{\theta})$ always increases except for $\Gamma(\tilde{y}, \tilde{\theta}) > \frac{|\hat{R}(t)|}{R} - 1$, but in this case is lower bounded by a positive constant. Moreover, since in this case we also have

$$
\begin{aligned}
\sigma \dot{\sigma} &= -\sigma \sin(\tilde{\theta}) \cos(\tilde{\theta}) \frac{R}{|\hat{R}(t)| - \tilde{y}} \frac{u}{R} \\
&= |\sigma| \, |\sin(\tilde{\theta})| \, |\cos(\tilde{\theta})| \frac{u}{|\hat{R}(t)| - \tilde{y}} > 0.
\end{aligned}
$$

the sliding manifold is not attractive. Car's trajectories may leave region 4 either through the boundary with region 1 $\tilde{\theta} = 0$ with $\tilde{y} > -R$, or by jumping symmetrically to region 3 for a change of sign of $\hat{R}(t)$.

In conclusion we can state the following

Proposition 1 *The kinematic model of a Dubin's car (2), subjected to the control law (16), converges to a reference path satisfying assumption A), B), C), provided that its initial configurations are within a suitable neighborhood of the origin.*

Remark 1 For the special case where the reference path is a straight line, i.e. $|\hat{R}(s)| \equiv +\infty$, the motion along the sliding manifold $\sigma = 0$ corresponds to motions of the unicycle along circles of radius R tangent to the reference path. The phase of reaching the sliding manifold results in a motion on a circle of radius R that brings the unicycle on the nearest circle tangent to the reference path. Furthermore, it can be proved in this case that the control law (16) ensures asymptotic stabililty of the reference path within an attractive domain $|\tilde{y}| < 2R$, $|\tilde{\theta}| < \pi$.

Remark 2 If the forward velocity is non constant but any given strictly positive function $u(t) \leq \alpha > 0$, the method above can be suitable modified by setting

$$
\omega = \text{sign}(\hat{R}(t)) \, \text{sign}(\sigma) \frac{u(t)}{R}.
$$

The prof of proposition 1 can easly be exented by simply rescaling time to the lenght of the path traveled by the car, i.e. $t = \int_0^t u(\tau) d\tau$, (see Sampei [23] for a related technique).
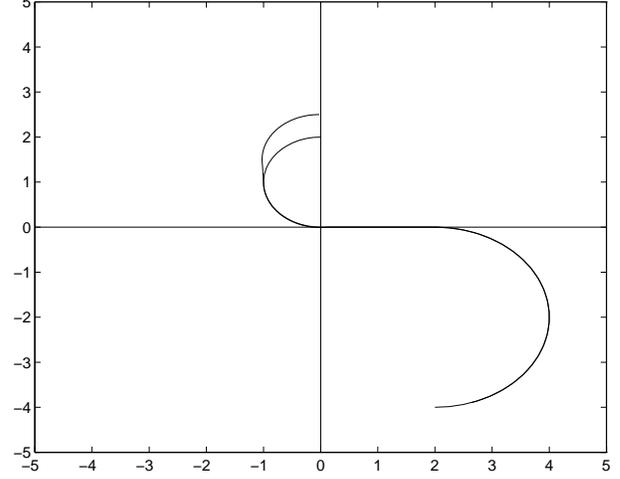

Figure 3: Reference path and motion of the unicycle.

4 Simulation

Let the reference path to be followed (see fig. 3.) be described as

$$
\hat{x}(s) = \begin{cases} -\sin(4\pi s) & 0 \leq s < 0.25 \\ 4(s - 0.25) & 0.25 \leq s < 0.75 \\ 2 + 2\sin(4\pi(s - 0.75)) & 0.75 \leq s \leq 1 \end{cases}
$$

$$
\hat{y}(s) = \begin{cases} 1 + \cos(4\pi s) & 0 \leq s < 0.25 \\ 0 & 0.25 \leq s < 0.75 \\ -2 - 2\cos(4\pi(s - 0.75)) & 0.75 \leq s \leq 1, \end{cases}
$$

Since the given path is comprised of a half–circle of radius 1, a segment of lenght 2 and a half–circle of radius 2, it meets the hypotheses A), B) and C). We set the forward velocity $u = 1$ and the minimum radius of curvature $R = 1$. The following results are relative to the initial configuration

$$
\begin{pmatrix} x_0 \\ y_0 \\ \theta_0 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0 \\ \pi \end{pmatrix}
$$

In fig. 3 the motion of the unicycle in the world frame (dash–dot line) is compared with the reference path (solid line). Chattering in the control input, a typical phenomenon in VSC, can be eliminated by employing a "boundary layer" smoothing technique (Slotine and Sastry [2]). The vehicle motion corresponding to such smoothed version of the proposed technique is reported in fig. 3 (dotted line). In fig. 4 the smoothed control input signal ω is reported. In fig. 5, the trajectory on the reduced state space $(\tilde{y}, \tilde{\theta})$ is reported with the
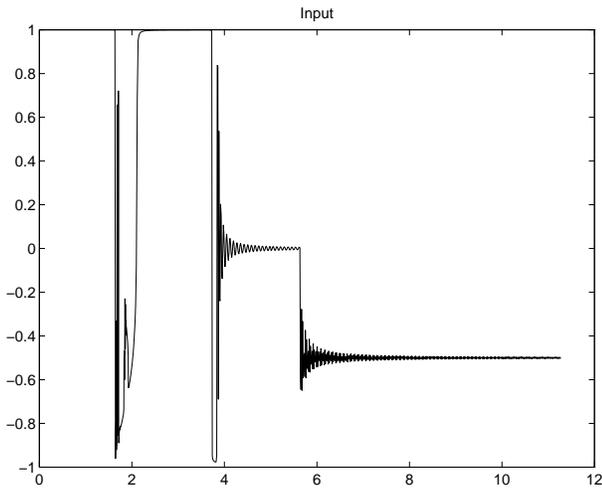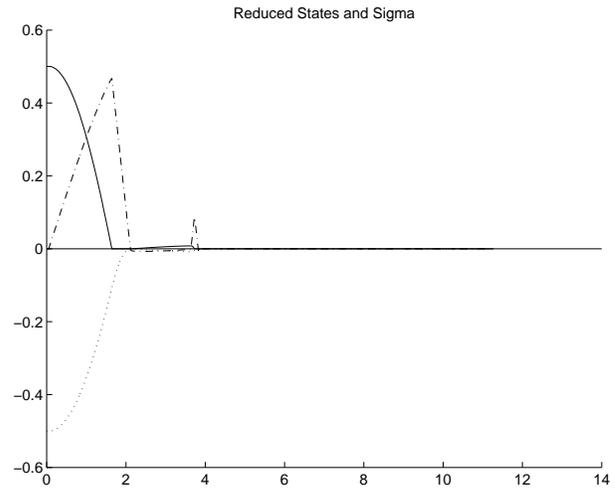
Figure 4: Input signals omega(t).

Figure 6: Evolution of the states y prime(t), theta prime(t) and of sigma(t).
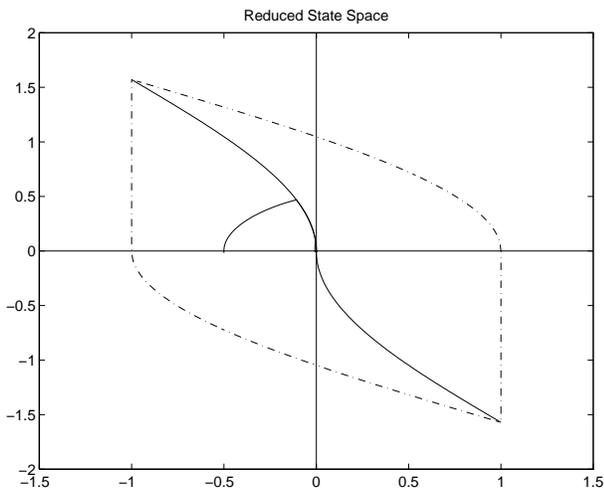

Figure 5: Trajectory on the reduced state space y prime, theta prime, domain of convergence and sliding manifold

domain of convergence and the sliding manifold $\sigma = 0$. Finally, in fig. 6 the evolution of the states $\tilde{y}(t), \tilde{\theta}(t)$ and $\sigma(t)$ is reported.

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